Online List Colorings with the Fixed Number of Colors

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Abstract

The online list coloring is a widely studied topic in graph theory. A graph G is 2-paintable if we always have a strategy to complete a coloring in an online list coloring of G in which each vertex has a color list of size 2. In this paper, we focus on the online list coloring game in which the number of colors is known in advance. We say that G is $[2, t]$-paintable if we always have a strategy to complete a coloring in an online list coloring of G in which we know that there are exactly t colors in advance, and each vertex has a color list of size 2.

Let $M(G)$ denote the maximum t in which G is not $[2, t]$-paintable, and $m(G)$ denote the minimum $t \geq 2$ in which G is not $[2, t]$-paintable. We show that if G is not 2-paintable, then $2 \leq m(G) \leq 4$, and $n \leq M(G) \leq 2n - 3$. Furthermore, we characterize G with $m(G) \in \{2, 3, 4\}$ and $M(G) \in \{n, n+1, 2n-3\}$, respectively.

1 Introduction

The concept of list coloring was introduced by Vizing [3] and by Erdős, Rubin, and Taylor [1]. For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A *k-list assignment* L of a graph G is a list assignment L such that $|L(v)| = k$ for each $v \in V(G)$. A proper coloring c such that $c(v) \in L(v)$ for each $v \in V(G)$ is said to be a *list coloring*.

Schauz [2] and Zhu [5] independently introduced an online variation of list coloring. On each round i, *Painter* sees only the *marked set* V_i of vertices whose lists contain the color i. Painter has to choose an independent subset X_i of V_i to get the color i. In the worst case, it can be viewed in the game setting that an adversary, called *Lister*, chooses V_i on each round i to prevent a coloring.

Let f be a function from $V(G)$ to the set of nonnegative integers. We say that G is *f-paintable* if Painter can guarantee coloring all vertices with no vertex v is marked more than $f(v)$ times. It can be viewed that $f(v)$ is the number of colors that is contained in the list of v. We write $f \cong k$ if $f(v) = k$ for each vertex v. When G is f-paintable and $f \cong k$, we say that G is k-paintable.

In this paper, we let (G, f) denote the game on a graph G with f as the aforementioned function. The game (G, f) *contains* (H, h) means that H is a subgraph of G and $h(v) = f(v)$ for each $v \in V(H)$.

Three particular functions f', f^* and f'' are defined as follows. The game (G, f') has $f'(v) = 2$ for each v except one vertex u which has $f'(u) = 1$. The game (G, f^*) is similar to

(G, f') except if there is a unique vertex u with degree 1, then we always assign $f'(u) = 1$ for this vertex. The game (P_n, f'') is played on a path P_n for $n \geq 2$ with $f''(v) = 2$ for each internal vertex v and $f''(u) = 1$ for each endpoint u.

In this paper, we focus on an online list coloring with the given number of rounds to play, or equivalently, the given number of colors that appear in all lists. Note that the level of information about the number of rounds (colors) plays important role for outlining a strategy.

In this version, Painter knows the number of colors in advance. It is reasonable to assume that Painter knows the number of colors in some applications. One maybe more interested in the "worst case version" of game for Painter, namely, Painter does not know the number of colors and Lister knows that Painter does not know the number of colors. It is certain that the study of the worst case version is more complicated. Nonetheless, the knowledge from the study on this version is possibly useful for facilitating the understanding of other variations.

We say that G is $[f, t]$*-paintable* if Painter guarantees to win in (G, f) with exactly t rounds. If G is $[f, t]$-paintable and $f \cong k$, then we call G is $[k, t]$*-paintable*. Let $M(G, f)$ denote the maximum t in which G is not $[f, t]$-paintable, and let $m(G, f)$ denote the minimum $t \geq \max\{f(v) : v \in V(G)\}$ in which G is not $[f, t]$-paintable. If no confusion arises, we may write $M(G)$ and $m(G)$ instead of $M(G, f)$ and $m(G, f)$ for $f \cong 2$.

The *remaining game* (G_i, f_i) after round i (where V_i and X_i are chosen) is defined recursively as follows. Let $(G_0, f_0) = (G, f)$. For $i \geq 1$, $(G_i, f_i) = (G, f_i)$ where $f_i(v) = f_{i-1}(v) - 1$ if $v \in V_i$, and $f_i(v) = f_{i-1}(v)$ if $v \notin V_i$. If a vertex v is in X_j for some $j \leq i - 1$, then we regard v to be colored already and v needs no coloring furthermore in (G_i, f_i).

Let $\theta_{p_1, p_2, \ldots, p_r}$ denote a graph obtained by identifying all beginnings and identifying all endpoints of r disjoint paths having $p_1, p_2, \ldots, p_r$ edges respectively. A path P_m and a cycle C_n intersect at one endpoint of P_m is denoted by $P_m \cdot C_n$. Two vertex disjoint cycles C_m and C_n connected by a path P_k is denoted by $C_m \cdot P_k \cdot C_n$. We always allow P_k in the notation to be P_1. The *core* of a graph G is the subgraph of G obtained by the iterated removal of all vertices of degree 1 from G.

Let $\mathfrak{F}_1 = \{C_{2n+1}\}$, $\mathfrak{F}_2 = \{C_m \cdot P_k \cdot C_n\}$, $\mathfrak{F}_3 = \{\theta_{p,q,r}$ that is not isomorphic to $\theta_{2,2,2n}\}$, $\mathfrak{F}_4 = \{\theta_{2,2,2n}$ that is not isomorphic to $\theta_{2,2,2}\}$, $\mathfrak{F}_5 = \{K_{2,n}$ where $n \geq 4\}$, and $\mathfrak{F} = \bigcup_{i=1}^{5} \mathfrak{F}_i$.

2 Preliminaries and Tools

Lemma 1 *Assume that G is not 2-paintable. A graph G is bipartite if and only if $m(G) \geq 3$.*

Proof. Note that for a game $(G, f \cong 2)$ with exactly two rounds, we have $V_1 = V_2 = V(G)$.

Necessity. Assume G is a bipartite graph with partite sets A and B. Since $V_1 = V_2 = V(G)$, Painter can choose $X_1 = A$ and $X_2 = B$ to complete a coloring. Thus $m(G) \geq 3$.

Sufficiency. Let $m(G) \geq 3$. In a game of two rounds, Painter can choose X_1 and X_2 which are independent sets to complete a coloring. Thus G is a bipartite graph with partite sets X_1 and X_2. $\square$

Lemma 2 *Let G be a disjoint union of graphs H and M. Let $f(v) = h(v)$ for $v \in V(H)$, and $f(v) = m(v)$ for $v \in V(M)$. If H is h-paintable and M is m-paintable, then G is f-paintable.*

Proof. We prove by induction on the number of uncolored vertices of G. Obviously, G is f-paintable if G has no uncolored vertices. For the induction step, assume that Lister chooses V_1 in the first round. If $V_1 \cap V(H) \neq \emptyset$, then Painter chooses X_1' that can counter $V_1 \cap V(H)$ in (H, h), otherwise Painter chooses $X_1' \neq \emptyset$. A set $X_2'' \subseteq V_1 \cap V(M)$ is chosen similarly. For (G, f), Painter chooses $X_1 = X_1' \cup X_1''$ to respond for V_1. The graph G_1 in the remaining game (G_1, f_1) is the disjoint union of two games that Painter can win. Moreover, (G_1, f_1) has fewer uncolored vertices than (G, f). By induction hypothesis, (G_1, f_1) is f_1-paintable. Thus G is f-paintable. □

In a digraph G, a set of vertices U is *kernel* of $V' \subseteq V(G)$ if U is an independent dominating set of $G[V']$.

Lemma 3 *If T is a tree, then T is f'-paintable.*

Proof. Let T be an n-vertex tree. It is clear that Painter wins when $n = 1$. Consider $n \geq 2$. Let u be a unique vertex with $f'(u) = 1$. Orient T into a digraph in which every vertex has in-degree 1 except u which has in-degree 0. In the first round, Painter chooses a kernel X_1 in V_1. Now, G_1 in the remaining game (G_1, f_1) is a forest in which each nontrivial tree has all of its vertex v satisfying $f_1(v) = 2$ except at most one vertex w with $f_1(w) = 1$. By induction hypothesis and Lemma 2, we have G_2 is f_1-paintable. □

Theorem 4 *An odd cycle C_n is not $[2, t]$-paintable if and only if $2 \leq t \leq n$.*

Proof. Consider a game $(C_n, f \cong 2)$ with exactly t rounds.

Necessity. Assume $t \geq n + 1$. Then (i) $V_1 \neq V(C_n)$, or (ii) $V_1 = V(C_n), |V_i| = 1$ for $2 \leq i \leq t = n + 1$, and $V_2 \cup \cdots \cup V_t = V(C_n)$. For (ii), Painter just greedily colors a vertex in V_i to win.

For (i), V_1 induces a union of disjoint paths. Orient $V(C_n)$ to be a directed cycle. In the first round, Painter chooses a kernel X_1 in V_1. Now, the set of uncolored vertices in (G_1, f_1) induces a union of paths in which each nontrivial path has all of its vertex v satisfying $f_1(v) = 2$ except at most one vertex u with $f_1(u) = 1$. By Lemmas 2 and 3, Painter has a winning strategy for the remaining game.

Sufficiency. Assume $2 \leq t \leq n$. Lister chooses $V_1 = V(C_n)$. Regardless of X_1, the remaining game $(G_1, f_1 \cong 1)$ has two adjacent vertices u and v which are uncolored. For $t = 2, \ldots, t-1$, Lister chooses V_i to be a set of one vertex other than u and v. Finally, in round t, Lister chooses V_t to contain each vertex w with $f_{t-1}(w) = 1$ (including u and v). The remaining game $(G_t, f_t \cong 0)$ has u or v uncolored. Thus C_n is not $[2, t]$-paintable for $2 \leq t \leq n$. □

Lemma 5 *Let the game (G, f) contains (H, h). Let $K = \max\{f(v) : v \in V(G) - V(H)\}$. If H is not $[h, t]$-paintable, then G is not $[f, k]$-paintable for $\max\{t, K\} \leq k \leq t + \sum_{v \in V(G)-V(H)} f(v)$. In particular, $m(G, f) \leq \max\{K, m(H, h)\}$ and $M(G, f) \geq M(H, h) + \sum_{v \in V(G)-V(H)} f(v)$.*

Proof. Lister can win (G, f) with $\max\{t, K\}$ rounds by using the strategy similar to one for (H, h) with t rounds, except that Lister also includes each vertex $v \in V(G) - V(H)$ in V_i for $i = 1, \ldots, f(v)$.

For $\max\{t, K\} + 1 \le k \le \sum_{v \in V(G)-V(H)} f(v)$, Lister has a winning strategy obtained from the above by moving vertices in $V(G) - V(H)$ to V_i for $i = \max\{t, K\} + 1, \ldots, k$ as needed. The remaining follows immediately. $\square$

Lemma 6 *$m(P_2, f'') = 1$ and $m(P_n, f'') = 2$ for $n \ge 3$.*

Proof. The result for $m(P_2, f'')$ is obvious. Consider $n \ge 3$. Let $V(P_n) = \{v_1, \ldots, v_n\}$. If n is even, then Lister chooses $V_1 = V(P_n) - \{v_1, v_n\}$. The remaining game (G_1, f_1) always has adjacent vertices u and v with $f_1(u) = f_1(v) = 1$. Lister chooses $V_2 = V(P_n)$ to win the game. If n is odd, then Lister chooses $V_1 = V(P_n) - \{v_n\}$. The remaining game (G_1, f_1) always has adjacent vertices u and v with $f_1(u) = f_1(v) = 1$. Lister chooses $V_2 = V(P_n) - \{v_1\}$ to win the game. $\square$

Lemma 7 *If G is a connected bipartite graph with a cycle, then $m(G, f') = 3$.*

Proof. Let u be a unique vertex with $f'(u) = 1$ in a connected bipartite graph G with a cycle C.

Consider a game (G, f') with two rounds. Let A and B be partite sets of G such that $u \in A$. Note that $V_1 = V(G)$ and $V_2 = V(G) - \{u\}$,or $V_1 = V(G) - \{u\}$ or $V_2 = V(G)$. If $u \in V_1$, then Painter chooses $X_1 = A$ and $X_2 = B$, otherwise Painter chooses $X_1 = B$ and $X_2 = A$. This makes Painter wins. Thus $m(G, f') \ge 3$.

Next, we show that $m(G, f') \le 3$. Let v be a vertex in C which is nearest to u. Note that u and v can be the same vertex. Lister chooses $V_1 = \{x, y\}$ where xy is an edge in $C - \{v\}$. Whatever X_1 is, the remaining game (G_1, f_1) contains (P_n, f'') for some $n \ge 2$. The remaining game is not $[f_1, 2]$-paintable by Lemmas 5 and 6. Thus $m(G, f') \le 3$. This completes the proof. $\square$

3 Finding $m(G)$

Lemma 8 *If G is bipartite and contains $H \in \mathfrak{F}_2$, then $m(G) = 3$.*

Proof. Lemma 1 yields $m(G) \ge 3$. Using Lemma 5, we only need to show that $C_m \cdot P_k \cdot C_n$ is not $[2, 3]$-paintable to show $m(G) \le 3$. First, Lister chooses $V_1 = \{v_1, v_2, w_1, w_2\}$, where v_1v_2 is an edge in C_n, w_1w_2 is an edge in C_m, and each vertex in V_1 is not a cut vertex. Regardless of X_1, the remaining game (G_1, f_1) contains (P_j, f'') for some $j \ge 3$. The remaining game is not $[f_1, 2]$-paintable by Lemmas 5 and 6. Thus $m(G) \le 3$. Lemma 1 yields $m(G) \ge 3$ which completes the proof. $\square$

Lemma 9 *If G is bipartite and contains $H \in \mathfrak{F}_3$, then $m(G) = 3$.*

Proof. Using Lemma 5, we only need to show that $\theta_{p,q,r}$ where $p, q \ge 3$, is not $[2, 3]$-paintable to show $m(G) \le 3$. Let $P = uw_1 \ldots w_{p-1}v$, $Q = ux_1x_2 \ldots x_{q-1}v$, and $R = uy_1y_2 \ldots y_{r-1}v$ be paths in $\theta_{p,q,r}$. First, choose $V_1 = \{w_1, w_2, x_1, x_2\}$. Regardless of X_1, the remaining game (G_1, f_1) contains (P_n, f'') for some $n \ge 3$. The remaining game is not $[f_1, 2]$-paintable by Lemmas 5 and 6. Thus $m(G) \le 3$. Lemma 1 yields $m(G) \ge 3$ which completes the proof. $\square$

Lemma 10 *If G is bipartite and contains $H \in \mathfrak{F}_4$, then $m(G) = 3$.*

Proof. Using Lemma 5, we only need to show that $\theta_{2,2,n}$ where $n \geq 4$, is not $[2,3]$-paintable to show $m(G) \leq 3$. Let $P = uav$, $Q = ubv$, and $R = ux_1x_2 \ldots x_{n-1}v$ be paths in $\theta_{2,2,n}$. In the first round, Lister chooses $V_1 = \{u, v, a, x_1, x_{n-1}\}$. Regardless of X_1, the remaining game (G_1, f_1) contains the game of (P_k, f'') for some $k \geq 3$. The remaining game is not $[f_1, 2]$-paintable by Lemmas 5 and 6. Thus $m(G) \leq 3$. Lemma 1 yields $m(G) \geq 3$ which completes the proof. □

Lemma 11 *If $G \in \mathfrak{F}_5$, then $m(G) = 4$.*

Proof. Let partite sets of G be $X = \{x_i : i = 1, 2, \ldots, n\}$ and $Y = \{u, v\}$. It is well known in the topic of list coloring that $K_{2,4}$ is not L-colorable if $L(u) = \{1, 2\}, L(v) = \{3, 4\}, L(x_1) = \{1, 3\}, L(x_2) = \{1, 4\}, L(x_3) = \{2, 3\}$, and $L(x_4) = \{2, 4\}$. Thus $m(K_{2,4}) \leq 4$. Lemma 5 yields $m(G) \leq 4$.

The winning strategy of Painter in the game of 3 rounds is as follows: Painter colors both u and v immediately after the first V_i that contains u and v, and greedily colors other legal vertices in other rounds. It can be seen that each vertex can be colored. Thus Painter wins in the game of 3 rounds. This concludes $m(G) = 4$.

□

Lemma 12 *Assume (G, f) contains (H, h) and H is a core of G.*
(a) If H is (h, t)-paintable and $2 \leq f(v) \leq t$ for each $v \in V(G) - V(H)$, then G is (f, t)-paintable.
(b) If H is h-paintable and $f(v) \geq 2$ for each $v \in V(G) - V(H)$, then G is f-paintable.

Proof. (a) We outline Painter's winning strategy for (G, f) as follows. Let F be the forest obtained from $G - E(H)$. Note that each tree T in F contains at most one vertex u in H. Suppose in round i, Lister chooses V_i. If $V_i \cap V(H) \neq \emptyset$, there is $X(H)_i$ to counter $V_i \cap V(H)$ in a game (H, h). Painter views a game in the part of each tree T in F as a game of (T, g) where $g(x) = f(x)$ for each $x \in V(T) - V(H)$ and $g(u) = 1$ for a unique vertex in $T \cap H$ (if exists.) For each tree T and round i, Painter considers the marked set $V(T)_i$ in the game (T, g) as $(V_i \cap V(T) - \{u\}) \cup (X(H)_i \cap V(T)$. Since $g(u) = 1$, Painter chooses u to be in the set $X(T)_i$ in round i if and only if $u \in X(H)_i$.

Since the coloring of vertices in $V(H)$ which depends on Painter's strategy in the game of (H, h) is a winning strategy, all vertices in H will be colored. By Lemma 3, all vertices in each T will be colored.

(b) is an immidiate consequence of (a). □

Lemma 13 *Suppose H is the core of a graph G and H contains a subgraph in $\mathfrak{F}_5$. Then*
(a) $H \in \mathfrak{F}_5$ or H contains a subgraph in $\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$,
(b) $m(G) = 4$ if and only if $H \in \mathfrak{F}_5$.

Proof. (a) Since H is 2-connected, we can create H from $K_{2,m}$ by successively adding ears (an ear is an edge or a path through new vertices connecting two existing vertices) or closed ears (a closed ear is a cycle with exactly one existing vertex). First, we grow $K_{2,m}$ to be the maximal subgraph $K_{2,n}$ in H. For this $K_{2,n}$, let u, v be in the same partite set and $a_1, \dots, a_n$ be in the other. If we cannot add more edges from this point, we have $H = K_{2,n}$. If we can add closed ear, then G contains $C_s \cdot P_1 \cdot C_t$. If the added ear connects u (or v) and a_i, then G contains $C_s \cdot P_1 \cdot C_t$. Consider the case that the added ear has the length q connecting u and v. By maximality of $K_{2,n}$, we have $q \neq 2$. Thus if q is odd, then H contains an odd cycle, otherwise H contains $\theta_{2,2,2t} \in \mathfrak{F}_4$. Consider the case that the added ear connects a_1 and a_2. Then the path obtained from the ear plus a_2v is an a_1v-path of length at least 3. This path together with a_1v and a_1ua_3v form $\theta_{1,3,q}$ where $q \geq 3$. This completes the first part.

(b) *Necessity.* Suppose the core H of a graph G contains a subgraph in $\mathfrak{F}_5$. By (a), $H = K_{2,n}$, or G contains $\bigcup_{i=1}^{4} \mathfrak{F}_i$. But the latter case implies $m(G) \leq 3$ by Lemmas 1, 8, 9, and 10. Hence $H = K_{2,n}$ where $n \geq 4$.

Sufficiency Suppose $H \in \mathfrak{F}_5$. Note that G is bipartite. Thus G is [2,2]-paintable by Lemma 1. Lemma 11 yields H is [2,3]-paintable but not [2,4]-paintable. Finally, Lemma 12 yields G is [2,3]-paintable and Lemma 5 yields G is not [2,4]-paintable. Hence $m(G) = 4$. □

Theorem 14 *[5] A graph G is 2-paintable if and only if the core of G is K_1, C_{2n}, or $K_{2,3}$. Equivalently, G is not 2-paintable if and only if the core of G contains a subgraph in $\mathfrak{F}$.*

Now we can classify $m(G)$ for each non-2-paintable graph G as follows.

Theorem 15 *Let G be a non-2-paintable graph. Then $m(G) = 2, 3,$ or 4. More specifically, we have*
(a) $m(G) = 2$ if and only if G is not bipartite,
(b) $m(G) = 3$ if and only if G is bipartite and contains a subgraph in $\mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$,
(c) $m(G) = 4$ if and only if G has a core in $\mathfrak{F}_5$.

Proof. The statement (a) is exactly Lemma 1. The statement (c) comes from Lemma 13. Let G be a non-2-paintable graph with the core H. By Theorem 14, H contains a subgraph in $\mathfrak{F}$. By (a) and (c), it remains to consider the case that G is bipartite and H is not in $\mathfrak{F}_5$. By Lemma 13, H contains a subgraph in $\mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$. Since G is bipartite, we have $m(G) \geq 3$. By Lemmas 5, 8, 9, and 11, we have $m(G) \leq 3$. Thus the remaining case satisfies both $m(G) = 3$ and G contains a subgraph in $\mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$. This completes the proof. □

4 On $M(G)$

Note that $\lg n = \log_2 n$.

Lemma 16 *For $n \geq 3$, (P_n, f'') is not $[f'', t]$-paintable if $2 \leq t \leq 2n - 2 - \lg n$.*

Proof. Let $n \geq 3$ and $V(P_n) = \{v_1, \ldots, v_n\}$. We show that (P_n, f'') is not $[f'', t]$-paintable for $2 \leq t \leq 2n - 2 - \lg n$ by induction. From Lemma 6, we know that (P_n, f'') is not $[f'', 2]$-paintable. Consequently, the desired statement is true for $n = 3$.

For $n \geq 4$, Lister begins with $V_1 = \{v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor + 1}\}$.

Consider the case $v_{\lfloor n/2 \rfloor} \notin X_1$. Then the remaining game (G_1, f_1) contains $(P_{\lfloor n/2 \rfloor}, f'')$. By induction and Lemma 5, the remaining game is not $[f_1, t]$-paintable for $2 \leq t \leq 2\lfloor n/2 \rfloor - 2 - \lg\lfloor n/2 \rfloor + 2\lceil n/2 \rceil - 2 = 2n - 3 - \lg(2\lfloor n/2 \rfloor)$. Thus the remaining game is not $[f_1, t]$-paintable for $2 \leq t \leq 2n - 3 - \lg n$. Including the first turn, Lister can win (P_n, f'') with t rounds for $3 \leq t \leq 2n - 2 - \lg n$.

Consider the case $v_{\lfloor n/2 \rfloor + 1} \notin X_1$. By induction and Lemma 5, the remaining game is not $[f_1, t]$-paintable for $2 \leq t \leq 2\lceil n/2 \rceil - 2 - \lg\lceil n/2 \rceil + 2\lfloor n/2 \rfloor - 2 = 2n - 3 - \lg 2\lceil n/2 \rceil$. Note that $\lfloor 2n - 3 - \lg 2\lceil n/2 \rceil \rfloor = \lfloor 2n - 3 - \lg n \rfloor$. Since t is an integer, the remaining game is not $[f_1, t]$-paintable for $2 \leq t \leq 2n - 3 - \lg n$. Including the first turn, Lister can win (P_n, f'') with t rounds for $3 \leq t \leq 2n - 2 - \lg n$.

□

Let $V(P_m) = \{x_1, \ldots, x_m\}, V(C_n) = \{v_1, \ldots, v_n\}$, and $P_m \cdot C_n$ be obtained from P_m and C_n by identifying v_n with x_1. Let (G, f^*) have $f^*(x_m) = 1$ and $f^*(v) = 2$ for each remaining vertex v. Note that m is allowed to be 1.

Lemma 17 *If $G = P_m \cdot C_n$, then*
(a) (G, f^*) is not $[f^*, 2]$-paintable if and only if n is odd,
(b) for $t \geq 3$, (G, f^*) is not $[f^*, t]$-paintable if $t \leq 2m + 2n - 4 - \lg(m + \lfloor n/2 \rfloor)$.

Proof. (a) *Necessity.* If n is even, then G is bipartite. Thus G is $[f^*, 2]$-paintable by Lemma 7.

Sufficiency. For n is odd, Lister chooses $V_1 = V(G)$. Then the remaining game (G_1, f_1) always contains adjacent uncolored vertices v and w in C_n such that $f_1(v) = f_1(w) = 1$. Next, Lister chooses $V_2 = V(G) - \{x_m\}$ to win the game. Thus (G, f^*) is not $[f^*, 2]$-paintable.

(b) Lister chooses $V_1 = \{v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor + 1}\}$.

If $v_{\lfloor n/2 \rfloor} \notin X_1$, then the remaining game (G_1, f_1) contains $(P_{\lfloor n/2 \rfloor + m}, f'')$ which is induced by $\{v_1, v_2, \ldots v_{\lfloor n/2 \rfloor}, x_1, x_2, \ldots, x_m\}$. By Lemmas 5 and 16, (G_1, f_1) is not $[f_1, t]$-paintable if $2 \leq t \leq 2(\lfloor n/2 \rfloor + m) - 2 - \lg(m + \lfloor n/2 \rfloor) + 2(\lceil n/2 \rceil - 1) - 1 = 2m + 2n - 5 - \lg(m + \lfloor n/2 \rfloor)$.

If $v_{\lfloor n/2 \rfloor + 1} \notin X_1$, then the remaining game (G_1, f_1) contains $(P_{\lceil n/2 \rceil + m - 1}, f'')$ which is induced by $\{v_{\lfloor n/2 \rfloor + 1}, v_{\lfloor n/2 \rfloor + 2}, \ldots, v_n = x_1, x_2, \ldots, x_m\}$. Thus (G_1, f_1) is not $[f_1, t]$-paintable if $2 \leq t \leq 2(\lceil n/2 \rceil + m - 1) - 2 - \lg(m + \lceil n/2 \rceil - 1) + 2\lfloor n/2 \rfloor - 1 = 2m + 2n - 5 - \lg(m + \lceil n/2 \rceil - 1)$. Thus (G_1, f_1) is not $[f_1, t]$-paintable if $2 \leq t \leq 2m + 2n - 5 - \lg(m + \lfloor n/2 \rfloor)$.

Thus, including the first round, we have (G, f^*) is not $[f^*, t]$-paintable if $t \leq 2m + 2n - 4 - \lg(m + \lfloor n/2 \rfloor)$. □

Note that the bound in Lemma 17 is not sharp if m is large.

Theorem 18 *Let G be a non-2-paintable graph with n vertices. Then*
(a) if $G = C_r \cdot P_k \cdot C_s$ with $r, s \geq 4$, then $M(G) \geq n + 2$,
(b) if $G = \theta_{p,q,r}$ and $p \geq 3, q + r \geq 4$, then $M(G) \geq n + 2$,
(c) if $G = K_{2,4}$, then $M(G) = n + 1 = 7$,
(d) $M(G) \geq n$,
(e) $M(G) = n$ if and only if G is an odd cycle.

Proof. (a) Consider $G = C_r \cdot P_k \cdot C_s$. Let $V(C_r) = \{v_1, \ldots, v_r\}$ and v_r be identified with an end vertex of P_k. Choose $V_1 = \{v_{\lfloor r/2\rfloor}, v_{\lfloor r/2\rfloor+1}\}$. If $v_{\lfloor r/2\rfloor} \notin X_1$, then the remaining game (G_1, f_1) contains $(C_s \cdot P_{k+\lfloor r/2\rfloor}, f^*)$. By Lemmas 5 and 17, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2(k + \lfloor r/2\rfloor) + 2s - 4 - \lg(k + \lfloor r/2\rfloor + \lfloor s/2\rfloor) + 2(\lceil r/2\rceil - 1) - 1 = 2k + 2r + 2s - 7 - \lg(k + \lfloor r/2\rfloor + \lfloor s/2\rfloor)$.

If $v_{\lfloor r/2\rfloor+1} \notin X_1$, then the remaining game (G_1, f_1) contains $(C_s \cdot P_{k+\lceil r/2\rceil-1}, f^*)$. By Lemmas 5 and 17, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2(k + \lceil r/2\rceil - 1) + 2s - 4 - \lg(k + \lceil r/2\rceil - 1 + \lfloor s/2\rfloor) + 2\lfloor r/2\rfloor - 1 = 2k + 2r + 2s - 7 - \lg(k + \lceil r/2\rceil - 1 + \lfloor s/2\rfloor)$. Thus (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2k + 2r + 2s - 7 - \lg(k + \lfloor r/2\rfloor + \lfloor s/2\rfloor)$.

Note that $|G| = k + r + s - 2$. Thus, including the first round, we have $M(G) \ge 2k + 2r + 2s - 6 - \lg(k + \lfloor r/2\rfloor + \lfloor s/2\rfloor) = 2n - 2 - \lg(k + \lfloor r/2\rfloor + \lfloor s/2\rfloor) \ge n + 2$. Note that the last inequality comes from $k \ge 1, r \ge 4$, and $s \ge 3$.

(b) Consider $G = \theta_{p,q,r} \in \mathfrak{F}_3 \cup \mathfrak{F}_4$. Let $P = uw_1 \ldots w_{p-1}v$, $Q = ux_1x_2 \ldots x_{q-1}v$, and $R = uy_1y_2 \ldots y_{r-1}v$ be paths in $\theta_{p,q,r}$. Choose $V_1 = \{w_{\lfloor p/2\rfloor}, w_{\lfloor p/2\rfloor+1}\}$. If $w_{\lfloor p/2\rfloor} \notin X_1$, then the remaining game (G_1, f_1) contains $(C_{q+r} \cdot P_{\lfloor p/2\rfloor+1}, f^*)$. By Lemmas 5 and 17, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2\lfloor p/2\rfloor + 2 + 2(q + r) - 4 - \lg(\lfloor p/2\rfloor + 1 + \lfloor (q + r)/2\rfloor) + 2(\lceil p/2\rceil - 1) - 1 = 2p + 2q + 2r - 5 - \lg(\lfloor p/2\rfloor + 1 + \lfloor (q + r)/2\rfloor)$.

If $w_{\lfloor p/2\rfloor+1} \notin X_1$, then the remaining game (G_1, f_1) contains $(C_{q+r} \cdot P_{\lceil p/2\rceil}, f^*)$. By Lemmas 5 and 17, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2\lceil p/2\rceil + 2(q + r) - 4 - \lg(\lceil r/2\rceil - 1 + \lfloor (q + r)/2\rfloor) + 2\lfloor p/2\rfloor - 1 = 2p + 2q + 2r - 5 - \lg(\lceil p/2\rceil + \lfloor (q + r)/2\rfloor$. Thus (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2p + 2q + 2r - 5 - \lg(\lfloor p/2\rfloor + 1 + \lfloor (q + r)/2\rfloor)$.

Note that $|G| = p + q + r - 1$. Thus, including the first round, we have $M(G) \ge 2p + 2q + 2r - 4 - \lg(\lfloor p/2\rfloor + 1 + \lfloor (q + r)/2\rfloor) = 2n - 2 - \lg(\lfloor p/2 + 1\rfloor + \lfloor (q + r)/2\rfloor) \ge n + 2$. Note that the last inequality comes from $p \ge 3$ and $q + r \ge 4$.

(c) Let partite sets of G be $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{a, b\}$. Observe that Lister has to choose $V_1 = \{a, x_1, x_2\}$ (or the set of vertices inducing P_3) to win the game. If $x_1, x_2 \notin X_1$, then (G_1, f_1) contains (P_3, f'') which is induced by $\{x_1, b, x_2\}$. By Lemmas 5 and 16, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2(3) - 2 - \lg 3 + f_1(a) + f_1(x_3) + f_1(x_4) = 9 - \lg 3$. Note that $9 - \lg 3 \ge 7$.

If $a \notin X_1$, then (G_1, f_1) contains $(C_4 \cdot P_1, f'')$ which is induced by $\{a, b, x_3, x_4\}$. By Lemmas 5 and 17, (G_1, f_1) is not $[f_1, t]$-paintable for $3 \le t \le 2(1+4) - 4 - \lg 3 + f_1(x_1) + f_1(x_2) = 8 - \lg 3$. Note that $8 - \lg 3 \ge 6$.

Including the first turn, we have $M(G) \ge 7 = |G| + 1$.

(d) By Theorem 14, the core of G contains a subgraph $H \in \mathfrak{F}$. Lemma 5 yields $M(G) \ge M(H) + 2(n - |H|)$. From (a), (b), (c), and Theorem 4, $M(H) \ge |H|$. Thus $M(G) \ge 2n - |H| \ge 2n - n = n$.

(e) *Necessity.* In the proof of (d), $M(G) = n$ only if $M(H) = |H| = n$. From From (a), (b), (c), and Theorem 4, H is an odd cycle C_n. If $H \ne G$, then G contains a smaller odd cycle C_m with $m < n$. Using the proof in (d), $M(G) \ge 2n - m \ge n + 2$ which is a contradiction. Thus G is an odd cycle.

The Sufficiency part is an immediate consequence of Theorem 4. $\square$

Assume G is a non-f-paintable graph. Let $q(G, f)$ be the minimum value for $\sum(|V_i| - 1)$ that Lister guarantees to have where each V_i is a set of marked vertices leading to an *uncolorable* vertex (that is an uncolored vertex v with $f_j(v) = 0$ for some j) with a restriction

that each vertex v is in at most $f(v)$ sets of V_is.

For example, consider the game (P_3, f'') where v_1 and v_3 be endpoints of the path and v_2 be the remaining vertex. Suppose Lister chooses $V_1 = \{v_1, v_2\}$. If Painter does not color v_1, then v_1 becomes an uncolorable vertex. But we cannot conclude that $q(G, h) = |V_1| - 1 = 1$ because Painter may color v_1. Painter can choose $V_2 = \{v_2, v_3\}$ to guarantee an uncolorable vertex in any cases. Thus we can conclude that $q(C_3, h) \leq 2 = (|V_1| - 1) + (|V_2| - 1)$. Lister can continue to choose $V_3 = \{v_3\}$ but this does not affect the value of $\sum(|V_i| - 1)$ and an uncolorable vertex is still uncolorable. Generally, if $V_1, \ldots, V_k$ guarantee to force an uncolorable vertex, then Lister can choose each remaining V_i to be singleton to retain the value of $\sum(|V_i| - 1)$. It can be seen that this process is unnecessary to continue for finding $q(G, f)$.

Similarly, if $V_1 = \{u\}$, then Painter can color u. This does not lead to an uncolorable vertex and the value $|V_1| - 1 = 0$ does not affect the value of summation. Thus we assume that V_i is not a singleton until an uncolorable vertex occurs. If $f(v) = 2$ for each $v \in G$, we just write $q(G)$ instead of $q(G, f)$. The next Lemma shows the relation of $q(G, f)$ and $M(G, f)$. For convenience, we use $\sum f(v)$ instead of $\sum_{v \in V(G)} f(v)$.

Lemma 19 $M(G, f) = \sum f(v) - q(G, f)$.

Proof. Since Lister can win in a painting game with $M(G, f)$ rounds, Lister can make marked sets $V_1, \ldots, V_{M(G,f)}$ to win a game in which each vertex v is in exactly $f(v)$ sets of V_is. Note that $\sum(|V_i| - 1) = \sum f(v) - M(G, f)$. Since $q(G, f)$ is the minimum value of $\sum(|V_i| - 1)$ leading to an uncolorable vertex, we have $q(G, f) \leq \sum f(v) - M(G, f)$. Thus $M(G, f) \leq \sum f(v) - q(G, f)$.

Next, by definition of $q(G, f)$, Lister can make marked sets $V_1, \ldots, V_k$ to force an uncolorable vertex with $q(G, f) = \sum_{i=1}^{k}(|V_i| - 1)$. After that Lister can choose each V_i for $i = k + 1, \ldots, k + \sum f_k(v)$ to be a singleton to complete the game (G, f). Since Painter cannot color an uncolorable vertex, Lister wins by this strategy. Consider $\sum f_k(v) = \sum f(v) - \sum_{i=1}^{k} |V_i| = \sum f(v) - \sum_{i=1}^{k}(|V_i| - 1) - k = \sum f(v) - q(G, f) - k$. That is Lister can win (G, f) with $k + \sum f_k(v) = \sum f(v) - q(G, f)$ rounds. Thus $M(G, f) \geq \sum f(v) - q(G, f)$. This completes the proof. □

Lemma 19 implies that finding $q(G, f)$ leads to knowing $M(G, f)$. If Painter forces an uncolorable vertex after choosing $V_1, \ldots, V_k$, Painter can minimize $\sum(|V_i| - 1)$ by choosing V_j to be a singleton for each $j \geq k+1$. But a singleton V_j contributes 0 in $\sum(|V_i| - 1)$. Thus to find $q(G, f)$, we may stop counting when an uncolorable vertex occurs.

Next we investigate the condition that $q(G, h) = 0, 1, 2$, or 3 where each vertex v has $h(v) = 1$ or 2.

Lemma 20 *No graph G satisfies $q(G, h) = 0$.*

Proof. To achieve $q(G, h) = 0$, each marked set V_i is a singleton. All vertices can be colored which is a contradiction. □

Lemma 21 $q(G, h) = 1$ *if and only if* (G, h) *contains* (P_2, f'').

Proof. *Necessity.* Let $q(G, h) = 1$. Then there is a marked set $V_1 = \{a, b\}$ forcing an uncolorable vertex. If a and b are not adjacent, then Painter can color both vertices. If $h(a) = 2$, then Painter can color b. In both situations, an uncolorable vertex does not occur which is a contradiction. Thus a and b are adjacent with $h(a) = 1$. Similarly, $h(b) = 1$. Thus (G, h) contains (P_2, f'').

Sufficiency. Assume (G, h) contains (P_2, f''). By Lemma 20, $q(G, f) \geq 1$. It remains to show that $q(G, f) \leq 1$. Choosing V_1 that induces (P_2, f''). we have $|V_1| - 1 = 1$ and V_1 forces an uncolorable vertex. This completes the proof. $\square$

We say that a set of vertices $A = \{v_1, v_2, \ldots, v_k\}$ in (G, f) induces (H, h) if A induces the graph H and $f(v_i) = h(v_i)$ for each i.

Lemma 22 *$q(G, h) = 2$ if and only if (G, h) does not contain (P_2, f''), but contains (P_3, f''), (P_4, f''), or $(C_3 \cdot P_1, f')$.*

Proof. *Necessity.* Let $q(G, h) = 2$. If (G, h) contains (P_2, f''), then $q(G, h) = 1$ by Lemma 21 which is a contradiction. To have $\sum(|V_i| - 1) = q(G, h) = 2$, (i) Lister can choose V_1 with size 3 forcing an uncolorable vertex, or (ii) Lister can chooses V_1 and V_2, each of which has size 2, forcing an uncolorable vertex.

Consider (i). Since (G, h) does not contain (P_2, f''), Painter can color each $v \in V_1$ satisfying $h(v) = 1$. An uncolorable vertex does not occur. Thus situation (i) is impossible.

Consider (ii). Let $V_1 = \{a, b\}$. If a and b are not adjacent, then Painter can color both a and b. Then V_2 must induce (P_2, f'') to force an uncolorable vertex which is a contradiction. Thus a and b are adjacent.

For $h(a) = 1$, we assume that Painter chooses $X_1 = \{a\}$, otherwise Lister can choose V_2 to be any 2-set to have $\sum(|V_i| - 1) = 2$ and an uncolorable vertex. Consider the remaining game (G_1, f_1). Thus $q(G_1, f_1) = 1$. By Lemma 22, (G_1, f_1) contains (P_2, f''). Since (G, h) does not contain (P_2, f''), this (P_2, f'') contains a vertex b. Moreover, there is a vertex $c \neq a$ which has $h(c) = 1$ and is adjacent to b. Since (G, h) does not contain (P_2, f''), we have a and c are not adjacent. Thus (G, h) contains (P_3, f'') induced by $\{a, b, c\}$.

Consider the case $h(a) = h(b) = 2$. Since $q(G, h) = 2$, the remaining game (G_1, f_1) always has $q(G_1, f_1) = 1$ regardless of X_1. By Lemma 21, (G_1, f_1) contains (P_2, f''). Thus if $a \notin X_1$, then there is c, an adjacent vertex of a, such that $\{a, c\}$ induces (P_2, f''). This also implies $h(c) = 1$. Similarly, there exists a vertex d which has $h(d) = 1$ and is adjacent to b, If $c \neq d$, then (G, h) contains (P_4, f'') induced by $\{a, b, c, d\}$. If $c = d$, then (G, h) contains (C_3, f') induced by $\{a, b, c\}$.

Sufficiency. Assume G does not contain (P_2, f''). Lemmas 20 and 21 imply $q(G, h) \geq 2$. It remains to prove $q(G, h) \leq 2$.

Suppose $\{v_1, v_2, v_3\}$ induces (P_3, f''). Then $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_2, v_3\}$ force an uncolorable vertex.

Suppose $\{v_1, v_2, v_3, v_4\}$ induces (P_4, f''). Then Painter chooses $V_1 = \{v_2, v_3\}$. If $v_2 \notin X_1$, then $V_2 = \{v_1, v_2\}$ forces an uncolorable vertex. If $v_3 \notin X_1$, then $V_2 = \{v_3, v_4\}$ forces an uncolorable vertex.

Suppose $\{v_1, v_2, v_3\}$ induces (C_3, f') where $f'(v_1) = 1$. Then Painter chooses $V_1 = \{v_2, v_3\}$. If $v_2 \notin X_1$, then $V_2 = \{v_1, v_2\}$ forces an uncolorable vertex. If $v_3 \notin X_1$, then $V_2 = \{v_1, v_3\}$ forces an uncolorable vertex.

In each case, We have $\sum(|V_i|-1)=2$ and an uncolorable vertex. Thus $q(G,h)\leq 2$ which completes the proof. $\square$

Lemma 23 *G contains C_3 if and only if $q(G)=3$.*

Proof. *Necessity.* Let $V(C_3)=\{a,b,c\}$. Lister chooses $V_1=\{a,b,c\}$. Since Painter can color at most one vertex, we may assume a and b are not colored. Choosing $V_2=\{a,b\}$ forces an uncolorable vertex. Thus $q(G)\leq 3$. From Lemma 20, 21, and 22, we have $q(G)\geq 3$. Thus the equality holds.

Sufficiency. Consider the choice of V_1 that makes $\sum(|V_i|-1)=3$ and leads to an uncolorable vertex. Since we want $\sum(|V_i|-1)=2$, we have $|V_1|\leq 4$. If $|V_1|=4$, then remaining V_is are singletons. Thus V_1 must force an uncolorable vertex. But $f(v)=2$ for each vertex v, an uncolorable vertex does not occur. Thus $|V_1|\neq 4$.

Consider $V_1=\{a,b\}$. Assume that Painter chooses $X_1=\{a\}$. By Lemma 22, the remaining game (G_1,f_1) contains (P_3,f''), (P_4,f''), or (C_3,f'). Since $f(v)=2$ for each vertex v, we have (G_1,f_1) contains $(C_3\cdot P_1,f')$ and $b\in C_3$. Thus G contains C_3.

Consider $V_1=\{a,b,c\}$. If a is not adjacent to b, then Painter can choose $X_1=\{a,b\}$. By Lemma 21, the remaining game (G_1,f_1) must contain (P_2,f''). This is possible only if c is adjacent to a vertex v with $f(v)=1$. But $f(v)=2$ for each vertex v. This is a contradiction. Thus each pair of vertices in V_1 are adjacent, that is G contains C_3. $\square$

Theorem 24 *Let G be a non-2-paintable graphs with n vertices. Then the followings hold:*
(a) $M(G)\leq 2n-3$ for each graph G,
(b) $M(G)=2n-3$ if and only if G contains C_3.

Proof. (a) Suppose $M(G)\geq 2n-2$. Thus $q(G)\leq 2$ by Lemma 19. But this contradicts to Lemmas 20, 21, and 22.

(b) *Necessity.* $M(G)=2n-3$. By Lemma 19, $q(G)=3$. Thus G contains C_3 by Lemma 23.

Sufficiency. Assume that G contains C_3. We have $M(C_3)=3$ by Theorem 4. Using Lemma 5, we have $M(G)\geq 2n-3$. Combining with (a), we have the desired equality. $\square$

5 Further Investigation

Using Theorems 18 and 24, we have the following corollary.

Corollary 25 *If an n-vertex graph G is not 2-paintable, then $n\leq M(G)\leq 2n-3$.*

Moreover, we characterizes graphs with $M(G)=n$ and graphs with $M(G)=2n-3$. We turn our attention to find the characterizations of G with other values of $M(G)$.

Lemma 26 *If n is even, then $M(C_{n-1}\cdot P_2)=n+1$.*

Proof. Let $G = C_{n-1} \cdot P_2$. By Theorem 18, we have $M(G) \geq n+1$.

Next we show $M(G) \leq n+1$. Suppose Lister can win in a game of t rounds where $t \geq n+2$. Then (i) $V_1 \not\subseteq V(C_{n-1})$, or (ii) $V_1 = V(C_{n-1})$ and $|V_i| = 1$ for $2 \leq i \leq t = n+2$. Note that in (ii), $V_2 \cup \cdots \cup V_t = V(C_n)$. Thus Painter just greedily colors a vertex in V_i to win.

For (i), V_1 induces a union of disjoint trees. Let $v \in V(G) - V(C_{n-1})$ and u be a neighbor of v. Orient $V(C_{n-1})$ to be a directed cycle and $u \to v$. In the first round, Painter chooses a kernel X_1 in V_1. Now, the set of uncolored vertices in (G_1, f_1) induces a union of trees in which each tree has all of its vertex v satisfying $f_1(v) = 2$ except at most one vertex u with $f_1(u) = 1$. By Lemmas 2 and 3, Painter has a winning strategy for the remaining game. Thus $M(G) \leq n+1$ which completes the proof. $\square$

Theorem 27 *$M(G) = n+1$ if and only if G is $K_{2,4}$ or a 4-vertex graph containing C_3, or a core of G is an odd cycle C_{n-1}.*

Proof. *Necessity.* Let G be a non-2-paintable graph with $M(G) = n+1$. By Theorem 14, G has a subgraph $H \in \mathfrak{F}$. Choose such H with the minimum number of edges. By Lemma 5 and Theorem 18, we have $n+1 = M(G) \geq M(H) + 2(n - |H|) \geq |H| + 2(n - |H|) = 2n - |H|$. Thus $|H| \geq n-1$, that is $|H| = n$ or $n-1$.

Consider $|H| = n$. Suppose H is not bipartite. If H is not an odd cycle, then G contains $H' \in \mathfrak{F}$ such that $e(H') < e(H)$ which contradicts to the choice of H. Thus H is an odd cycle. Moreover $G = H$ since G cannot have an odd cycle smaller than H. But then $M(G) = n$ by Theorem 4 which is a contradiction. Thus H is bipartite. This implies H is the graph described in (a), (b), or (c) of Theorem 18. But if H is a graph in (a) or (b), then $M(G) \geq M(H) \geq n+2$. Thus $H = K_{2,4}$. If $G \neq K_{2,4}$, then G contains C_3 which again contradicts to the choice of H. Thus $G = K_{2,4}$.

Consider $|H| = n-1$. Suppose H is not an odd cycle. Then H is the graph described in (a), (b), or (c) of Theorem 18. By Lemma 5 and Theorem 18, $M(G) \geq M(H) + 2 \geq (|H| + 1) + 2 = n + 2$ which is a contradiction. Thus H is an odd cycle. Moreover H is an induced subgraph of G, otherwise H contains a smaller odd cycle which contradicts to the choice of H. If $H = C_3$, then G is a 4-vertex graph with C_3. Consider the case that H is an odd cycle with length at least 5. Let $v \in V(G) - V(H)$. If $\deg(v) \geq 2$, then $\deg(v) = 2$ and $G = \theta_{2,2,n-3}$, otherwise G has an odd cycle smaller than H, a contradiction. By Theorem 18 (b), $M(G) \geq n+2$, a contradiction. Thus $\deg(v) \leq 1$. This implies H is an odd cycle with $n-1$ vertices and it is a core of G.

Sufficiency. If $G = K_{2,4}$, then $M(G) = 7 = n+1$ by Theorem 18. If G is 4-vertex graph containing C_3, then $M(G) = 5 = |n+1$ by Theorem 24. If a core of G be an odd cycle C_{n-1}, then $M(G) = n+1$ by Lemma 26. $\square$

6 Remarks and Open Problems

Proceeding to characterize G with $M(G) = n+2$ is more involved. First, we need to analyze $M(H)$ where $H \in \mathfrak{F}$ more deliberately. Moreover, one has to consider the case $|H| = n-2$ and other cases carefully.

Meanwhile, the process to characterize G with $M(G) = 3$ can be applied to the characterization of G with $M(G) = 2n - 4$. First, begin by characterizing (G, h) with $q(G, h) = 3$, and then proceed to characterize G with $q(G) = 4$. However, the process is clumsy because many more cases arise.

Thus we propose the first problem.

Problem 1: Find the efficient method to characterize G with $M(G) = n+k$ or $M(G) = 2n-k$ for each k.

Assume that we know a graph G has $m(G) = 2$ and $M(G) = 2n - 3$. Is it true that G is not $[2, t]$-paintable for $2 \leq t \leq 2n - 3$? The answer is yes. By Theorems 15 and 24, G contains C_3. Using Lemma 5, we have G is not $[2, t]$-paintable for $2 \leq t \leq 2n - 3$. This motivates us to ask the second problem.

Problem 1: Suppose that G is not either $[f, t_1]$-paintable or $[f, t_2]$-paintable where $t_1 < t_2$. Is it true that G is not $[f, t]$-paintable if $t_1 < t < t_2$?

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